

The Walnut Digital Signature AlgorithmTM Specification

Iris Anshel, Derek Atkins, Dorian Goldfeld, and Paul Gunnells

SecureRF Corporation

100 Beard Sawmill Rd #350, Shelton, CT 06484

ianshel@securerf.com, datkins@securerf.com, dgoldfeld@securerf.com, pgunnells@securerf.com

Abstract. This document specifies the Walnut Digital Signature Algorithm (WalnutDSA), a submission to the NIST Post-Quantum Cryptography standardization process. WalnutDSA is a group-theoretic signature system based on non-abelian group theory combined with matrices, permutations, and arithmetic over finite fields. The computation imbalance has signature verification significantly more efficient than signature generation, allowing fast verification even on constrained 16- and 8-bit processors. Key and signature sizes are comparable to efficient non-quantum-resistant methods, and are much smaller than most other quantum-resistant methods.

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1 Introduction

The Walnut Digital Signature Algorithm (WalnutDSA) was introduced in 2016 [1] based on research for a lightweight signature method useful for constrained and embedded devices. The foundation of WalnutDSA is E -Multiplication, a one-way function published in 2005, which combines infinite group theory in a non-abelian group, matrices, permutation, lookups, and finite field arithmetic. Specifically, the group used by WalnutDSA is the Braid group. See [1] for a full security evaluation and additional references, including a proof of security under EUF-CMA.

In WalnutDSA a private key and a signature are braids, and a public key is a pair of matrices and permutations, and an ordered list of entries in the finite field called T-values. To generate a signature one takes the message to be signed, hashes it, converts the hash output to a braid, and then builds the signature. To verify a signature one performs the same hashing and braid-conversion of the message, then two sets of E -Multiplication operations, one matrix multiplication, and then compares the results.

E -Multiplication is extremely lightweight, easily implemented, and runs very efficiently even in small environments. This makes signature validation very efficient even on small, 16- and 8-bit processors. In lightweight hardware an E -Multiplication can occur in a single clock cycle.

This document specifies how to implement WalnutDSA including key generation, signature generation, and signature validation. It also includes the necessary statements to meet the requirements of the NIST Post-Quantum Standardization Process section 2.B, modulo the attached references (e.g., [1]).

2 Definitions

- B_N : The braid group on N strands.
- **Braid**: A configuration of N woven strands. The braid group is denoted B_N .
- **Braid Generators**: The braid group B_N has $N-1$ Artin generators labeled b_1, b_2, \dots, b_{N-1} which, along with their inverses (b_i^{-1}) can be used to define any braid.
- **CB Matrix**: A colored Burau matrix. c.f. [12].
- **Cloaking Element**: A braid of a special form that disappears during E -Multiplication. A cloaking element v is said to cloak for a permutation σ when $(m, \sigma) \star v = (m, \sigma)$.

- **Concealed Cloaking Element**: A cloaking element where the cloaked permutation is unknown.
- **E-Multiplication**: An action on a matrix, permutation, and braid resulting in a new matrix and permutation. The action is denoted by the symbol \star .
- **\mathbb{F}_q** : A finite field with q elements. In the case of a binary field, the polynomial elements of the field are translated directly into the binary integer representation.
- **Matrix**: An $N \times N$ grouping of elements. Herein these elements are computed over a finite field \mathbb{F}_q .
- **Permutation**: A one-to-one function from a set onto itself.
- **Purebraid**: A non-trivial braid whose corresponding permutation is trivial.
- **\mathcal{R}** : A braid rewriting function that maps a braid to another equivalent braid, used to obfuscate the structure.
- **σ_w** : A permutation of the braid word w .
- **T-values**: An array of N nonzero elements $\{\tau_1, \tau_2, \dots, \tau_N\}$ in the finite field \mathbb{F}_q .

3 Design Rationale

During the construction of WalnutDSA we encountered several decision points. The following are potential questions about WalnutDSA that may arise and the answers as to why the choices were made. Note that understanding these issues may require reading to the end and then returning to this section.

- Why are cloaking elements needed?

Cloaking elements exist to prevent WalnutDSA from being susceptible to the Conjugacy Search Problem.

The general goal was to produce a group element, using the signer's private key and the message, so that extracting the signer's private key would be algorithmically hard. The second goal was to use E-Multiplication for the verification step. The third goal was that the public key of the signer and some other readily available data facilitates one side of the verification. In order to bring the message into the signature, a hash function is applied and then a method for encoding the hash output into the braid group is performed. Since simply conjugating the encoded message by the signer's private key would not be sufficient to keep the private key secure, additional braids are inserted into the conjugate to obscure the private key. In order to facilitate (efficient) verification, these additional braids must not be too long and must satisfy the cloaking property.

- Why do we set $\tau_2 = -\tau_1^{-1}$?

In order to produce cloaking elements which are reasonably short, we need to set constraints on T-values. Specifically, if $\tau_1 \cdot \tau_2 = -1$ then by taking $b_i^{\pm 4}$ we end up with a short element that disappears. By evaluating the precise form the CB pair associated with the fourth power of Artin generators take (and conjugating appropriately) we see that the assumption $\tau_2 = -\tau_1^{-1}$ ensures that the conjugate of the fourth power will disappear when E-Multiplied as needed.

- Why are hashed messages encoded into a free subgroup of the pure braid subgroup?

We need to ensure that two distinct hashes of messages encode in distinct ways, i.e., yield different braids. We do this by associating with each possible block of bits in a hash value with a positive power of a braid which is a generator of a free subgroup of the braid group. Such expressions are necessarily distinct.

- Why are $N \geq 8$ and $q \geq 32$?

We need to ensure that a brute force attack on the method is ineffective. By choosing $N \geq 8$ and $q \geq 32$, we ensure there are sufficiently many public keys, and that the underlying algorithmically hard problems are robust. Moreover, we need to ensure that, for a k -bit security level, $q^{N(N-3)-1} > 2^{2k}$, given the dimension of the computed vector space from the encoded messages, $q^{\text{dimension}} > 2^{2k}$, and also $q^{N/2-1}\sqrt{60} > 2^k$.

- Why recommend b_i^{-4} for cloaking elements?

When generating a cloaking element, we can choose several options for the exponent. When q is prime we use a 4^{th} power, and when q is GF(2) we can choose 2 or 4. In either case, however, we can choose either a positive or negative exponent. The benefit of a negative exponent is that, when using BKL for rewriting, the BKL First Pass will collect all negative exponents to the beginning of the braidword. This implies that a cloaking element using a -4 exponent will, under BKL, mix more than a +4 exponent. We do not suggest always using a negative exponent because this mixing is only ensured under BKL.

4 Key Generation

First generate the private key:

1. Choose an integer $N \geq 8$ and associated braid group B_N .
2. Choose a finite field \mathbb{F}_q of $q \geq 32$ elements.
3. Compute the values $\kappa = 6\kappa_1 + 2\kappa_2$ from the security level, which determines the minimal number of concealed cloaking elements to include:

$$\kappa = \lceil \text{SecurityLevel} / \log_2(N!) \rceil$$

4. Compute the value ℓ from the security level, which determines the minimal length of the private key. To compute ℓ you need to solve the equation:

$$\ell = \text{Security Level} + \log_2(\ell) - \log_2(N - 1) - \log_2\left(\binom{\ell - 2 + N}{N - 1}\right)$$

Note that these four items are pre-defined by the security level parameters in Section 11 so in general nothing must be done to “choose” N , q , κ_1 , κ_2 , or ℓ .

5. Choose a random set of T-values = $\{\tau_1, \tau_2, \dots, \tau_N\}$, where each τ_i is an invertible element in \mathbb{F}_q ($\tau_i \neq 0, 1$), and then set $\tau_2 = -\tau_1^{-1}$ ensuring $\tau_2 \neq 1$. If $\tau_2 = 1$ then randomly choose a new τ_1 and recompute. Repeat until $\tau_2 \neq 1$.
6. Generate a random braid of length ℓ (see section 9). This braid must not be a purebraid, so regenerate if the permutation is trivial.
7. Freely reduce this braid (see section 5.1).
8. Generate a second random braid of length ℓ and freely reduce it. This braid must not be a purebraid, and must not have the same permutation as the first braid, so regenerate if the permutation is trivial or matches the permutation of the first braid.
9. Priv(S) is the first freely reduced braid, which has permutation σ_{PrivS} , and Priv(S') is the second freely reduced braid, which has permutation $\sigma_{PrivS'}$.

Next, compute the public key by E -Multiplication (\star):

1. $\text{Pub}(S) = (\text{Id}_N, \text{Id}_{S_N}) \star \text{Priv}(S)$, where Id_N is the $N \times N$ identity matrix and Id_{S_N} is the identity permutation in S_N . See section 7 for more information on E -Multiplication.
2. $\text{Pub}(S') = (\text{Id}_N, \text{Id}_{S_N}) \star \text{Priv}(S')$
3. Publish the public key with the following data:
 - N
 - q
 - T-values
 - $\text{Pub}(S)$
 - Matrix Part of $\text{Pub}(S')$

5 Signature Generation

The WalnutDSA method requires the full message to be hashed before being passed into the signature function. An appropriate hash method should be used and the output passed to the signature generator as the message \mathcal{M} . Suggested hash functions are denoted in the appropriate parameter selections (see Section 11).

To generate a signature:

1. Generate the encoded message $E(\mathcal{M})$ as per section 5.1.
2. Generate cloaking elements v_1, v_2, v_3 as defined in section 5.2 with κ_1 concealed cloaking elements (see section 5.3) on each side, where v_1 cloaks for the identity, v_2 cloaks for $\sigma_{\text{Priv}S'}$, and v_3 cloaks for $\sigma_{\text{Priv}S}$.
3. Add κ_2 concealed cloaking elements to $\text{Priv}(S)^{-1}$ and $\text{Priv}(S')$ – labeled $\kappa(S)$ and $\kappa(S')$. Note that σ_0 for $\kappa(S)$ is $\sigma_{\text{Priv}S}$, and that σ_0 for $\kappa(S')$ is the identity permutation. See section 5.3.
4. Compute the signature $\text{Sig} = \mathcal{R}(v_3 \cdot \kappa(S) \cdot v_1 \cdot E(\mathcal{M}) \cdot \kappa(S') \cdot v_2)$, a braid rewritten as per 5.4.

5.1 Message Encoding

The original input gets hashed and the output of the hash is the input to the WalnutDSA Message Encoder. We assume the length of the hash output \mathcal{M} is $2d$, a multiple of 2 bits long.

The encoding method utilizes the collection of pure braid generators given by the following equations [2]:

$$\begin{aligned}
x_1 &= g_{(N-1),N} = b_{N-1}^2 \\
x_2 &= g_{(N-2),N} = b_{N-1} \cdot b_{N-2}^2 \cdot b_{N-1}^{-1} \\
x_3 &= g_{(N-3),N} = b_{N-1} b_{N-2} \cdot b_{N-3}^2 \cdot b_{N-2}^{-1} b_{N-1}^{-1} \\
x_4 &= g_{(N-4),N} = b_{N-1} b_{N-2} b_{N-3} \cdot b_{N-4}^2 \cdot b_{N-3}^{-1} b_{N-2}^{-1} b_{N-1}^{-1} \\
&\vdots \\
x_{N-1} &= g_{1,N} = b_{N-1} b_{N-2} \cdots b_2 \cdot b_1^2 \cdot b_2^{-1} b_3^{-1} \cdots b_{N-1}^{-1}
\end{aligned}$$

Next we specify a sequence of 4-tuples of generators, S , where each entry in the sequence, $S_i = \{g_{i_1}, g_{i_2}, g_{i_3}, g_{i_4}\}$ identifies unique generators, and where $g_1 = x_{N-1}$, $g_2 = x_{N-2}$, $\dots g_{N-1} = x_1$. This specification only specifies the i_x values, e.g. $S_i = \{1, 3, 5, 7\}$ implies $\{g_1, g_3, g_5, g_7\} = \{x_{N-1}, x_{N-3}, x_{N-5}, x_{N-7}\}$. The sequence S repeats once the end is reached.

The message encoder proceeds as follows:

1. Assume there are $|S|$ members in the encoding sequence S .
2. Break the message \mathcal{M} into d 2-bit blocks.
3. For each block (the i^{th} block):
 - (a) Choose the i^{th} entry in the sequence S , S_i . Because S repeats, if $i > |S|$ then choose the entry $i \bmod |S|$.
 - (b) Use the two bits of the block (x) to choose the generator g_{i_x} in the sequence entry S_i .
 - (c) Append g_{i_x} to the result.

For example, if $S = \{\{1, 3, 5, 7\}, \{2, 4, 6, 8\}\}$ and the hex hash output is 0x1234, this results in the output $g_1 g_4 g_1 g_6 g_1 g_8 g_3 g_2$
4. The encoded message $E(\mathcal{M})$ is the freely reduced product of these d block results.

Free reduction is a rewriting process that removes certain pairs of elements in the braid – namely, a generator b_i followed immediately by its inverse b_i^{-1} , or an inverse generator b_i^{-1} followed by the generator b_i . Any such pair of consecutive elements may be erased from the braid. For example, the free reduction of the braid $b_1 b_2 b_2^{-1} b_3$ is $b_1 b_3$.

5.2 Cloaking Elements

A cloaking element is a braid of a special form that disappears during E -Multiplication (see section 7). A cloaking element v is said to cloak for a permutation σ when $(m, \sigma) \star v = (m, \sigma)$.

To generate a cloaking element that cloaks permutation σ with κ concealed cloaking elements:

1. Pick a random integer $2 \leq i \leq N - 1$.
2. Pick a random bit, ± 1 , for the cloaking element exponent.¹
3. Compute the permutation preimages (a, b) for 1 and 2 in σ , i.e., a is the value σ takes to 1, and b is the value σ takes to 2: $a = \sigma^{-1}(1)$, $b = \sigma^{-1}(2)$.
4. Choose a random permutation σ_w of high order that moves $i \rightarrow a$, $i + 1 \rightarrow b$ (see section 8).
5. Generate a random braid using permutation σ_w (see section 9) and invert it (so the result has permutation σ_w), calling the result w .
6. Add κ concealed cloaking elements (see section 5.3) to each of w and w^{-1} where κ is the specific parameter for this cloaking element. The σ_0 for the concealed cloaking elements are this cloaking element's permutation parameter, σ , for inserting into w and $\sigma \cdot \sigma_w$ for inserting into w^{-1} . Call these results $\kappa(w)$ and $\kappa(w^{-1})$.
7. Compute the cloaking element $v = \kappa(w) \cdot b_i^{\pm 1} \cdot \kappa(w^{-1})$.

¹ When using a rewriting method like BKL, having a cloaking element to a negative power will result in better obscuration

5.3 Concealed Cloaking Elements

A concealed cloaking element is a cloaking element where the permutation being cloaked is unknown a priori.

To insert κ concealed cloaking elements in the braidword bw with initial permutation σ_0 :

1. If $\kappa = 0$ then return bw .
2. Pick a random offset in the braidword bw : $0 < \text{offset} < \text{length}(bw) - 1$ ²
3. Split bw into two subwords bw_l and bw_r , where there are *offset* Artin generators in bw_l .
4. Determine the permutation of bw_l : σ_{bw_l}
5. Generate a cloaking element (see section 5.2) CE for the permutation $\sigma_0 \cdot \sigma_{bw_l}$ (see section 8.1) with $\kappa = 0$
6. Create a new braidword: $bw_{new} = bw_l \cdot CE \cdot bw_r$
7. Repeat this process starting from step 1 with $bw = bw_{new}$ and $\kappa = \kappa - 1$

5.4 Rewriting

The signature braid must be rewritten to hide the form and protect the private key. There are an infinite number of equivalent braids which means it is computationally infeasible to determine the original from the rewritten version. We specify two methods to rewrite the signature (BKL Normal Form and Stochastic Rewriting), plus a third option that may be used to reduce the final length (Dehornoy Reduction).

5.4.1 BKL Normal Form

Birman–Ko–Lee (BKL) Normal Form was introduced in 1998 [3] as a canonical form for a braid. Every braid can be converted to BKL Normal Form, and every equivalent braid will result in the same BKL output. For example, the braids $b_1 b_2 b_1$ and $b_2 b_1 b_2$ would result in the same output after running through BKL.

Please reference Section 4 of [3] for the algorithm description.

5.4.2 Stochastic Rewriting

Stochastic Rewriting is a new method which is useful for smaller processors because it just involves random rewriting from lookup tables. The process is:

1. Freely reduce the braid if it has not already been reduced.
2. Convert the braid to “Y generators” and freely reduce the result.
3. Partition the braid into chunks of random sizes between 5-10 generators each³:
 - (a) Generate random numbers between the minimum (5) and the maximum (10) inclusive, and subtract each from the initial total until the running drops below the minimum.
 - (b) If the running total is zero then the partition has been found.

² We want to ensure there is always at least one generator to the left and one generator to the right of the concealed cloaking element.

³ Note that there are 2^{W-1} possible partitions of a word of length W if partitions could be of any size. Limiting to blocks of size 5-10 reduces that number. Still, reversing the process is hard.

- (c) Otherwise (running total is not zero), jump back to the position in the list where the running total was greater than minimum * maximum (50) and set the running total to this value.
- (d) Repeat until the running total becomes zero.
- 4. For each partition, choose a random offset into the partition (from the first to second-last).
- 5. Take the offset and offset+1 generators and look up a relation in the relation table (see Appendix A).
- 6. Replace those two generators with the relation (if one is found) using the Pair Replacement method described below; otherwise, do nothing for this partition.
- 7. Once you reach the last partition, freely reduce and then return to step 3 in order to repeat the process 3 times.
- 8. After the third repetition, convert back to Artin generators and freely reduce.

Convert Artin to Y Generators

Use the following process to convert a single Artin generator $b_k \in B_N$ to Y generators using the partition p of $N - 1$. If the Artin generator is an inverse, then invert the Y result. Converting a full word just involves iteration of this method. Note that the partition here is a static partition of $N - 1$.

```
// Create an array the same length as the number of partitions of N-1
r := array[length_of(p)+1]

// Initialize this array with the sum of the partitions
r[1] = 1
for i in (2..length_of(p)+1):
    r[i] = r[i-1] + p[i-1]

// Determine which partition contains k
j = 1
while(r[j] <= k):
    j++
j--;

// Build the response
u = k - r[j]
if (u < p[j] - 1):
    answer = {{r[j]+u, 1}, {r[j]+u+1, -1}}
else:
    answer = {{r[j]+u, 1}}

return answer
```

Convert Y to Artin Generators

Use the following process to convert a single Y generator $y_k \in B_N$ to Artin generators using the partition p of $N - 1$. If the Y generator is an inverse, then invert the Artin result. Converting a full word just involves iteration of this method. Note that the partition here is a static partition of $N - 1$.


```

// Create an array the same length as the number of partitions of N-1
r := array[length_of(p)+1]

// Initialize this array with the sum of the partitions
r[1] = 1
for i in (2..length_of(p)+1):
    r[i] = r[i-1] + p[i-1]

// Determine which partition contains k
j = 1
while(r[j] <= k):
    j++
j--;

// Build the response
u = k - r[j]
answer = {}
for i in (0..p[j]-u-1):
    answer = answer + {r[j]+u+i, 1}

return answer

```

Integer Partitions

A partition of an integer is an ordered list of integers that sum to the desired total. For example, the integer 20 can be partitioned into $\{10,10\}$, or $\{5,5,5,5\}$, or $\{7,8,5\}$, or $\{7,5,8\}$, or any other random split.

Pair Replacement

The partitioning of $N - 1$ not only defines the Artin to Y generator mappings but also the set of relations between pairs of generators. This list of relations is enumerated in Appendix A. Given a Y-word of length two, the process searches through the list of relations until a match is found (i.e, it finds a relation that contains the length-two word). Note that not all pairs of generators have relations, so there may not be a match.

Once a relation is found, the replacement is made by replacing the original pair with the inverse of the surrounding relation by inverting the part of the relation to the left of the subword, and then the part to the right of the subword. For example, if one is searching for the subword $\{1,1\}\{2,1\}$ and the relation that was found is $\{3,1\}\{2,1\}\{1,1\}\{2,1\}\{3,1\}$. The left part is $\{3,1\}\{2,1\}$ and the right part is $\{3,1\}$. Once you invert and concatenate these parts, then the replacement would be $\{2,-1\}\{3,-1\}\{3,-1\}$.

Note that if there are multiple matches for the pair within the relation then randomly choose one of the matches.

5.4.3 Dehornoy Reduction

Dehornoy Reduction is a method to reduce the size of a braid by finding and removing complex cancelations beyond single free reduction [8]. While solving the shortest word problem in the braid group is known to be NP-Hard, Dehornoy is the best-known method to reduce a braid to a minimal length. Applying Dehornoy is recommended; however, it may be applied “later”. For example, a lightweight processor may generate a signature and use Stochastic Rewriting, and then send that (long) signature to another, more powerful device, which can then run Dehornoy.

Please reference Section 4 of [8] for the algorithm description.

6 Signature Verification

Signature verification depends on first hashing the input using the same hash method as the signature generation to generate the hash output \mathcal{M} . Then to verify the signature:

1. Any signature longer than 2^{14} generators is immediately considered invalid.
2. Compute $E(\mathcal{M})$ as per section 5.1.
3. Evaluate $(M_1, \sigma_1) = (\text{Id}_N, \text{Id}_{S_N}) \star E(\mathcal{M})$, where Id_N is the $N \times N$ identity matrix and Id_{S_N} is the identity permutation in S_N .
4. Evaluate $(M_2, \sigma_2) = \text{Pub}(S) \star \text{Sig}$.
5. Compute the matrix multiplication $M_3 = M_1 \cdot \text{MatrixPart}(\text{Pub}(S'))$.
6. Compare M_2 and M_3 for equality. If $M_2 = M_3$, then the signature is valid.

7 *E*-Multiplication

The one-way function *E*-Multiplication is an action that starts with a matrix and permutation, a braid, and results in a new matrix and permutation. *E*-Multiplication is iterative, and by definition is applied one braid generator at a time. One can find closed formula for applying certain longer braid words.

The best way to explain the process is via pseudo-code. To compute a single *E*-Multiplication starting with a matrix m , permutation p , and braid generator b_i^e :

```
// compute the multiplication values for this generator based
// on the T-values, strand, and whether the generator is inverted
if e == 1:
    a = T[p[i]]
    b = -a
    c = 1
else: // e == -1
    a = 1
    b = -Inverse(T[p[i+1]])
    c = -b

// iterate down columns and matrix-multiply each value
if i != 0:
    for j in (1..N):
```

```

        m[j][i-1] += m[j][i] * a

for j in (1..N):
    m[j][i+1] += m[j][i] * c

for j in (1..N):
    m[j][i] *= b

// swap permutation based on the generator
temp = p[i]
p[i] = p[i+1]
p[i+1] = temp

```

To compute the E -Multiplication of a longer braid, one just iterates this process over the whole braid, reading from left to right.

8 Permutations

To generate a random permutation use the Fisher-Yates Shuffle [9]:

1. Start with the identity permutation of n elements.
2. Start with the last (1-indexed) offset, $i = n$.
3. Choose a random number $1 \leq j \leq i$.
4. Swap permutation elements i and j .
5. Iterate i down to 1.

If the desired permutation has additional constraints, those constraints can be applied after this process is complete. For example, if one needs to move $i \rightarrow a$, then one takes a randomly constructed permutation and modifies it by the following:

1. Find the permutation preimage of a . This is the offset o where the permutation value is a .
2. Swap the entries at o and i .

Note that all offsets and values are 1-indexed in this definition.

8.1 Multiplying Permutations

Parts of the WalnutDSA method require multiplying permutations. To multiply two permutations, σ_X and σ_Y , to generate $\sigma_X \cdot \sigma_Y$, use the following process:

1. Iterate i from $1 \dots N$
2. Look up the i^{th} entry in σ_Y (call it j)
3. Assign the i^{th} entry in the result with the j^{th} entry in σ_X
4. The result is the multiplication $\sigma_X \cdot \sigma_Y$

9 Braids

To generate a random braid word of length l :

1. Choose a random braid generator b_i , where $1 \leq i < N$.
2. Choose a random power, $\epsilon = \{-1, 1\}$.
3. Append b_i^ϵ to the braid word.
4. Iterate l times.
5. Freely reduce the result.

To generate a random braid $b(\sigma)$ from permutation σ :

1. Convert the permutation σ to a product of transpositions $t_1 \cdots t_r$:
 - (a) First write σ as a product of disjoint cycles $C_1 \cdots C_s$ where the last element of each C_i is the smallest number in the cycle.
 - (b) Order the cycles such that the last element of each C_i is in ascending order.
 - (c) Convert each cycle to a product of transpositions: if $C_i = (a_1, \dots, a_k)$, then $C_i = (a_1, a_2)(a_1, a_3) \cdots (a_1, a_k)$.
 - (d) Replace each C_i with its corresponding product of transpositions and flatten the list.
2. For each transposition t_i , generate a random braid $b(t_i)$ that produces it:
 - (a) Find the smallest element m and largest element M exchanged by the transposition t , i.e., $t = (m, M)$.
 - (b) Set $b(t_i)$ to be the identity braid.
 - (c) For $k = m$ to $M - 1$, replace $b(t_i)$ with $b(t_i) \cdot b_k^\epsilon$, where $\epsilon = \{-1, 1\}$ is randomly chosen.
 - (d) For $k = 2$ to $M - m$, replace $b(t_i)$ with $b(t_i) \cdot b_{M-k}^\epsilon$, where again $\epsilon = \{-1, 1\}$ is randomly chosen.
3. The result⁴ $b(\sigma)$ is the product $b(t_1) \cdots b(t_r)$.

9.1 Pure Braids

The pure braid subgroup of B_N is generated [10] by the set of $(N)(N-1)/2$ braids given by:

$$g_{i,j} = b_{j-1}b_{j-2} \cdots b_{i+1} \cdot b_i^2 \cdot b_{i+1}^{-1} \cdots b_{j-2}^{-1}b_{j-1}^{-1}, \quad 1 \leq i < j \leq N.$$

To create a pure braid generator of B_N :

1. Choose random numbers i, j : $1 \leq i < j \leq N$.
2. Choose a random exponent $\epsilon = \{-1, 1\}$.
3. Iterate $0 \leq k < j - i - 1$ and append b_{j-k-1} .
4. Append $b_i^{2\epsilon}$.
5. Iterate $0 \leq k < j - i - 1$ and append b_{i+k+1}^{-1} .

⁴ Note that this process generates a braid that has permutation σ^{-1} which is why it is inverted in Section 5.2.

10 Object Encodings

Throughout this document, indices are often 1-indexed. For example, in B_N the generators are labeled b_1, b_2, \dots . However, computers are better with 0-indexed numbers, arrays, and matrices, so the encodings are 0-indexed.

For all encodings, multi-byte numbers are encoded in network byte order (i.e., most significant byte first). For example, the decimal number 255 is encoded in hex as 00 FF, decimal 256 as 01 00.

Larger data objects like matrices, permutations, and braids are “bit packed” to reduce the effective transmission size. Bit packing also uses most-significant-bit first. When packing a matrix, the entries are encoded from 0 to $q - 1$ and packed across each row sequentially. The permutation is packed as a series of entries from 0 to $N - 1$.

Braids get encoded first with a 2-byte length (which is the number of generators)⁵, and then each generator is encoded with one bit for the sign and additional bits for the strand. For example, in B_8 packing each braid generator requires 4 bits. In B_8 , encoding the braid $b_1 b_2^{-1} b_3 b_4 b_5^{-1} b_6 b_7^{-1}$ would result in the hex 00 07 09 23 C5 E0.

10.1 Public Key

The public key contains the following data:

- **N**: an 8-bit unsigned integer.
- **q**: a 16-bit field (see below).
- **Header Padding**: Parameter-specific extra header padding. If specified by the parameter definition, this will be the specified number of bytes zero-byte padding.
- **T-values**: a packed array of N entries in \mathbb{F}_q . This results in $N \log_2(q)$ bits, which gets rounded up to the nearest byte (padded with 0-7 bits of zeros) as necessary. The parameter specifies the packing rules.
- **Pub(S) Matrix**: a possibly-packed matrix of $N \times (N - 1) + 1$ entries in \mathbb{F}_q . We know the last row of the matrix is always 0, except for the last entry of the last row, so those $N - 1$ entries are elided from the packing. This results in $(N^2 - N + 1) \log_2(q)$ bits which gets rounded up to the nearest byte (padded with 0-7 bits of zeros) as necessary. The parameter specifies the packing rules.
- **Pub(S') Matrix**: a possibly-packed matrix of $N \times (N - 1) + 1$ entries in \mathbb{F}_q . We know the last row of the matrix is always 0, except for the last entry of the last row, so those $N - 1$ entries are elided from the packing. This results in $(N^2 - N + 1) \log_2(q)$ bits which gets rounded up to the nearest byte (padded with 0-7 bits of zeros) as necessary. The parameter specifies the packing rules.
- **Pub(S) Permutation**: a packed array of N entries from 0 to $N - 1$. This results in $N \log_2(N)$ bits, which gets rounded up to the nearest byte (padded with 0-7 bits of zeros).

The q field is a 16-bit field that encodes the field parameter. It a “big-endian” bit-string that has the following format, a 2-bit flag and a 14-bit big-endian value:

Bits:	7	6	5	4	3	2	1	0	7	6	5	4	3	2	1	0
Value:	Flag		Value													

⁵ In all testing to date, signature braids have never exceeded 10,000 generators.

Where the Flag defines the Value as:

- 0 0 Direct Field Number
- 0 1 Mersenne Prime: $2^{Value} - 1$
- 1 0 Reserved
- 1 1 Reserved

10.2 Signature

A signature is just a braid, so it is encoded as a single packed braid as detailed at the start of this section. It has a 2-byte (16-bit) integer length (the number of generators) followed by the packed list of generators. Because each generator encodes into 4 bits, you can fit two generators into every byte. If you have an odd number of generators then the final byte is padded with zeros.

It does not matter which reduction method is used; in all cases the signature is converted to and encoded in Artin generators. Moreover, the 2-byte length field is sufficient because in all cases the maximum length seen experimentally is well below 65,000 generators. A long signature could be the result of an attempted attack and must be considered invalid.

11 Parameter Specifications

11.1 Security Level: 128

For a classical security level of 2^{128} (which, subject to Grover, results in a quantum-safe security level of 2^{64}), use the following parameters:

- $N = 10$
- $q = 2^{31} - 1$ (aka M31)
- $\kappa_1 = 1$
- $\kappa_2 = 0$
- $\ell = 124$
- Hash function: SHA2-256
- Encoder Sequence: $\{3,5,7,9\}$, $\{2,4,6,8\}$, $\{1,3,5,7\}$, $\{2,4,6,8\}$ (Dimension 82)
- Matrix and T-value padding is elided
- Header Padding: 1 byte

This results in a 256-bit message size, at least 2^{350} possible public keys that would need to be searched, as well as at least 2^{128} possible secret keys and cloaking elements. The public key is 6232 bits (including the N/q values). The private key is variable length and has a maximum length of 1240 bits (not including any markers as to N , individual braid lengths, or the security level).

Signatures are variable length, and the actual resulting length also depends on which rewriting method gets used. Experimentally we can determine the expected minimum, maximum, and average lengths (see Table 1).

Rewriting Method	Minimum	Mean	Maximum
BKL + Dehornoy	5790	10186.05	16530
Stochastic + Dehornoy	5540	9939.60	16310
Stochastic w/o Dehornoy	15860	19608.85	23670

Table 1: Experimentally determined 128-bit signature lengths (in bits)

11.2 Security Level: 256

For a classical security level of 2^{256} (which, subject to Grover, results in a quantum-safe security level of 2^{128}), use the following parameters:

- $N = 10$
- $q = 2^{61} - 1$ (aka M61)
- $\kappa_1 = 2$
- $\kappa_2 = 0$
- $\ell = 275$
- Hash function: SHA2-512
- Encoder Sequence: $\{3,5,7,9\}$, $\{2,4,6,8\}$, $\{1,3,5,7\}$, $\{2,4,6,8\}$ (Dimension 82)
- Matrix and T-value padding is elided
- Header Padding: 5 bytes

This results in a 512-bit message size, at least 2^{560} possible public keys that would need to be searched, as well as at least 2^{256} possible secret keys and cloaking elements. The public key is 12408 bits (including the N/q values). The private key is variable length with a maximum of 2750 bits (not including any markers as to N , individual braid lengths, or the security level).

Signatures are variable length, and the actual resulting length also depends on which rewriting method gets used. Experimentally we can determine the expected minimum, maximum, and average lengths (see Table 2).

Rewriting Method	Minimum	Mean	Maximum
BKL + Dehornoy	12100	19280.80	28520
Stochastic + Dehornoy	12640	19394.25	29600
Stochastic w/o Dehornoy	31170	38468.20	42620

Table 2: Experimentally determined 256-bit signature lengths (in bits)

11.3 Security Level: 20 – For Testing Purposes

In order to test an insecure version of WalnutDSA, we suggest a smaller version at a classical security level of 2^{20} by using the following parameters:

- $N = 8$
- $q = 16$
- $\kappa_1 = 1$

- $\kappa_2 = 0$
- $\ell = 25$
- Hash function: SHA1
- Encoder Sequence: $\{1, 3, 5, 7\}$ (Dimension 14)
- Matrix and T-value packing is in effect

This results in a 160-bit message size, at least 2^{160} possible public keys that would need to be searched, as well as at least 2^{40} possible secret keys and cloaking elements. The public key is 544 bits (including the N/q values). The private key is variable length with a maximum of 200 bits (not including any markers as to N or the security level).

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A Stochastic Rewriting “Y” Generator Relations

The following is a list of 1-indexed Y relations in B_{10} with partition $\{5,4\}$ for use in the Stochastic Rewriting process, followed by a list of relations in B_8 with partition $\{4,3\}$. These lists are created by enumerating all braid relations available in Artin generators (e.g., $b_1 b_2 b_1 = b_2 b_1 b_2$, etc, and $b_1 b_3 = b_3 b_1$, etc) and converting them to Y generators, and also the additional relations that are available due to the change over to Y generators. Note that not every possible pair of $y_i y_j$ is in the list. This list is then sorted by length, which results in the following lists:

A.1 Relations in B_{10}

$\{5,1\}\{9,1\}\{5,-1\}\{9,-1\}$
 $\{9,-1\}\{5,1\}\{9,1\}\{5,-1\}$
 $\{9,1\}\{5,1\}\{9,-1\}\{5,-1\}$
 $\{2,1\}\{1,1\}\{5,1\}\{1,-1\}\{1,-1\}$
 $\{3,1\}\{1,1\}\{5,1\}\{2,-1\}\{1,-1\}$
 $\{3,1\}\{2,1\}\{5,1\}\{2,-1\}\{2,-1\}$
 $\{4,1\}\{2,1\}\{5,1\}\{3,-1\}\{2,-1\}$
 $\{4,1\}\{3,1\}\{5,1\}\{3,-1\}\{3,-1\}$
 $\{5,1\}\{3,1\}\{5,1\}\{4,-1\}\{3,-1\}$
 $\{7,1\}\{6,1\}\{9,1\}\{6,-1\}\{6,-1\}$
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 $\{7,-1\}\{6,1\}\{6,1\}\{9,-1\}\{6,-1\}$

$\{8, -1\}\{6, 1\}\{7, 1\}\{9, -1\}\{6, -1\}$
 $\{8, -1\}\{7, 1\}\{7, 1\}\{9, -1\}\{7, -1\}$
 $\{9, -1\}\{7, 1\}\{8, 1\}\{9, -1\}\{7, -1\}$
 $\{9, -1\}\{8, 1\}\{8, 1\}\{9, -1\}\{8, -1\}$
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A.2 Relations in B_8

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$\{6, -1\}\{3, 1\}\{4, -1\}\{6, 1\}\{7, -1\}\{4, 1\}\{3, -1\}\{7, 1\}$
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B Expected Computational Efficiency and Performance

When analyzing the computational efficiency of WalnutDSA, one must look at four distinct processes: key-pair generation, raw signature generation, signature rewriting, and signature validation.

A note on notation: when declaring the expected computation efficiency of the various subprocesses of WalnutDSA, the basis of the order is explicitly used when available. For example, using “ $\mathcal{O}(N)$ ” implies a linear operation in N , which is the number of strands of the braid in B_N , whereas using “ $\mathcal{O}(n)$ ” is a generic linear operation.

Performance was tested on a Linux server configured with 8 cores of Intel Xeon X5355 at 2.66GHz running at 2660237000 cycles per second and 32 GB RAM. The test code was compiled using: `gcc -march=native -mtune=native -O3 -fomit-frame-pointer -fwrapv -msse2`.

B.1 Key-Pair Generation

Key-pair generation is straightforward. The private key is purely a randomly generated braid, which takes $\mathcal{O}(\ell)$ operations to create and freely reduce. Generating the T-values is also an $\mathcal{O}(N)$ operation (but N is small). Finally, computing the public key via E -Multiplication requires N multiplies and $2N$ additions repeated ℓ times, so it is still $\mathcal{O}(N\ell)$ (remembering that N is fixed, so really $\mathcal{O}(\ell)$). We expect this to take fractions of a millisecond on the target platform. We also expect this function to be fast even on small processors because it is purely a function of the speed of the random number generator.

As shown in Figures 1 and 2, key generation on our test platform takes between 1.06-1.17 million cycles for a 128-bit key and 2.8-3.05 million cycles for a 256-bit key. This equates to 2273-2509 and 872-950 keys generated per second.

B.2 Raw Signature Generation

Raw signature generation is the process of taking the hash of the input message, converting the hash to a braid, generating cloaking elements, and putting the “raw” signature together prior to rewriting. We separate this from the rewriting portion because there are multiple rewriting methods proposed with different performance profiles.

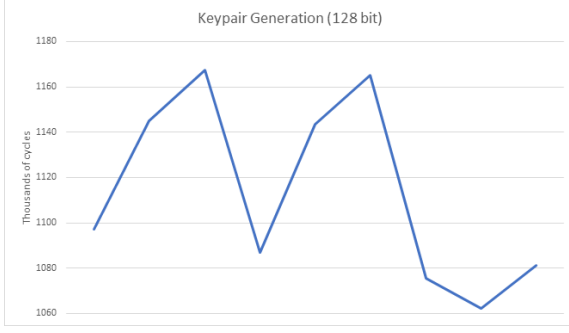


Fig. 1: Timing for generating 128-bit keys

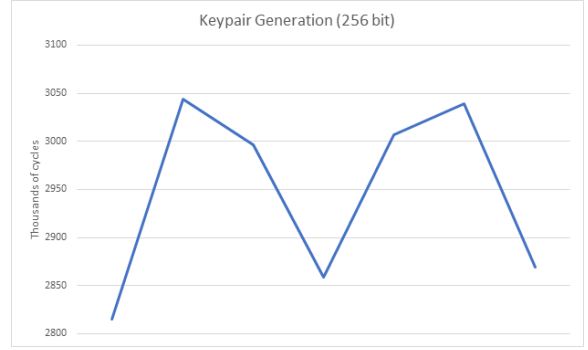


Fig. 2: Timing for generating 256-bit keys

Hashing the message is an $\mathcal{O}(n)$ operation in the length of the input message and is out of the control of WalnutDSA. We expect a good implementation of SHA2-256 or SHA2-512 to behave appropriately.

After hashing, we convert the hash output to a braid (this is an $\mathcal{O}(n)$ operation in the size of the hash output), and generate the three sets of cloaking elements (each an $\mathcal{O}(L)$ operation). Finally, we invert the private key ($\mathcal{O}(\ell)$) and put it all together ($\mathcal{O}(n)$). Of all these operations, the hash function is the most computationally intensive. The rest of the operations are purely limited on the speed of the random number generator. We expect this operation to be fast even on tiny devices.

B.3 Signature Rewriting

Rewriting the signature is the most computationally intensive operation in WalnutDSA, although it is required for signature security. There are three rewriting options:

1. BKL + Dehornoy
2. Stochastic Rewriting
3. Stochastic Rewriting + Dehornoy

The BKL algorithm, which outputs the canonical form of any braid, runs in $\mathcal{O}(n^2)$ time in the length of the input braid. BKL will convert any equivalent braid into the exact same output braid, making it easy to detect “sameness.” Of course the canonical form of a braid is often much longer than the original.

Enter Dehornoy, which takes a braid and shortens it by finding ways to manipulate the braid to remove inverses, even if they are not adjacent. The Dehornoy algorithm also runs in $\mathcal{O}(n^2)$ time in the length of the input braid.

However, both BKL and Dehornoy run in statistical time, not fixed time. Depending on the inputs they can complete very quickly or run somewhat longer. Moreover, the output is variable in length based on the inputs, which implies that WalnutDSA signatures are not constant length.

Running our implementation through SUPERCOP, we generated three sets of keys and then for each key ran 32 runs for each of 48 different message sizes (see Figure 3). At 128-bit security the signature generation took between 190-250 million cycles. The variability in execution time is due to the varying lengths of signatures across different inputs, how the hash

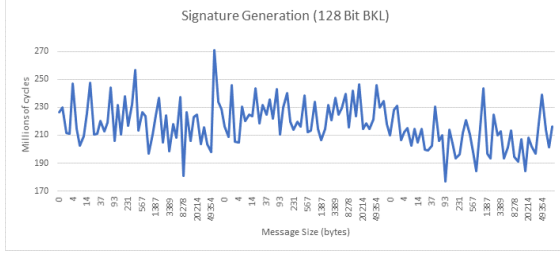


Fig. 3: SUPERCOP output for generating 128-bit signatures with BKL and Dehornoy

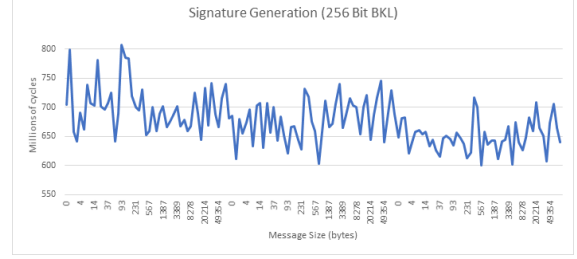


Fig. 4: SUPERCOP output for generating 256-bit signatures with BKL and Dehornoy

output gets converted into varying lengths of braids, and how that interacts with the random cloaking elements.

For 256-bit security (see Figure 4), the execution time increased to 600-800 million cycles.

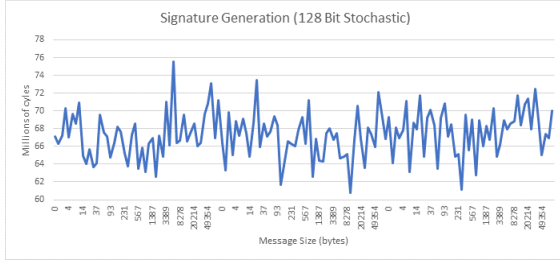


Fig. 5: SUPERCOP output for generating 128-bit signatures with Stochastic Rewriting and Dehornoy

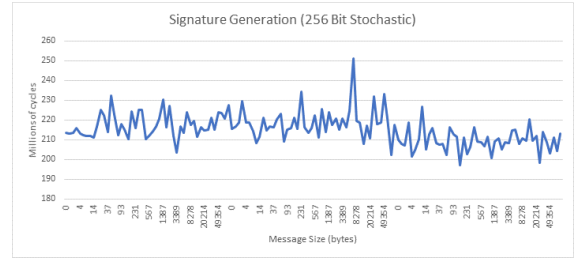


Fig. 6: SUPERCOP output for generating 256-bit signatures with Stochastic Rewriting and Dehornoy

The Stochastic Rewriting method is a mostly-linear operation that randomly replaces sections of a braid using a known set of braid relations. Its running time is slightly greater than linear, because the length of the braid increases on every round. The exact complexity is greater than $\mathcal{O}(n)$ but less than $\mathcal{O}(n \log(n))$.

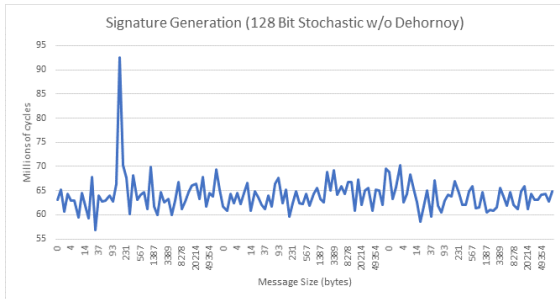


Fig. 7: SUPERCOP output for generating 128-bit signatures with Stochastic Rewriting without Dehornoy

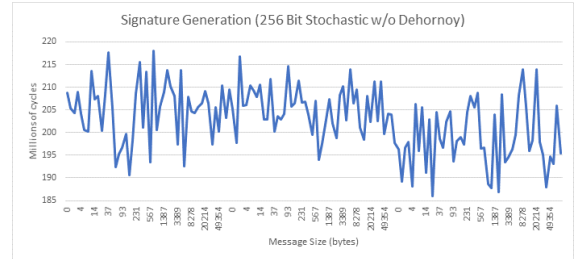


Fig. 8: SUPERCOP output for generating 256-bit signatures with Stochastic Rewriting without Dehornoy

As shown in Figures 5 and 6, replacing BKL with Stochastic Rewriting provides a significant speed increase. With the three keys randomly chosen at the 128-bit security level, signatures generated in 62-74 million cycles.

At the 256-bit security level speed is also increased. Those keys signed messages in 200-240 million cycles. Using Stochastic Rewriting shows a 2.5-3x speed improvement over BKL.

Moreover, it's likely that Stochastic Rewriting could be implemented on an embedded device. However in this case it's more likely that the embedded device would only run Stochastic Rewriting, then send the signature over to a more powerful device (trading of signature size and transmission time for computation capability). The larger device could run Dehornoy and reduce the signature for storage.

When you remove Dehornoy, our test system was able to generate a 128-bit signature in 60-70 million cycles (see Figure 7), and a 256-bit signature in 190-215 million cycles (see Figure 8). The complexity reduction is such that an embedded device may be sufficient; however, the resulting signature is longer.

B.4 Signature Validation

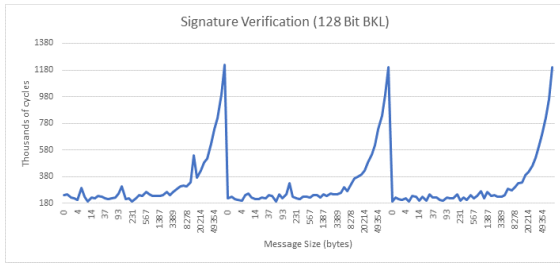


Fig. 9: SUPERCOP output for verifying 128-bit signatures with BKL and Dehornoy

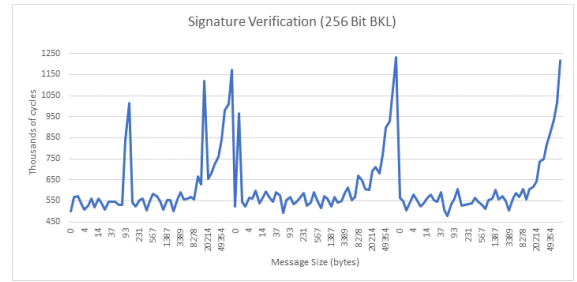


Fig. 10: SUPERCOP output for verifying 256-bit signatures with BKL and Dehornoy

Validating a signature requires hashing the message, converting the hash output to a braid ($\mathcal{O}(n)$), two sets of E -Multiplication ($\mathcal{O}(n)$ in the length of the signature and the length of the converted hash output), one matrix multiplication ($\mathcal{O}(N^3)$), and one matrix comparison ($\mathcal{O}(N^2)$).

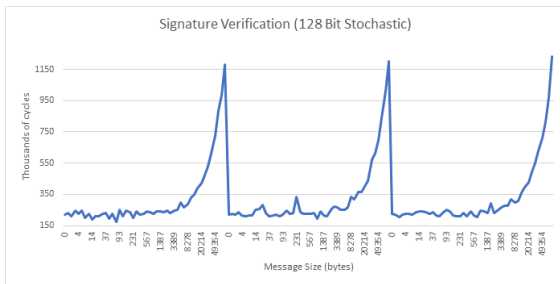


Fig. 11: SUPERCOP output for verifying 128-bit signatures with Stochastic Rewriting and Dehornoy

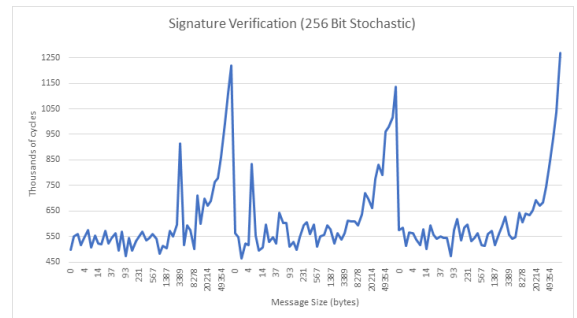


Fig. 12: SUPERCOP output for verifying 256-bit signatures with Stochastic Rewriting and Dehornoy

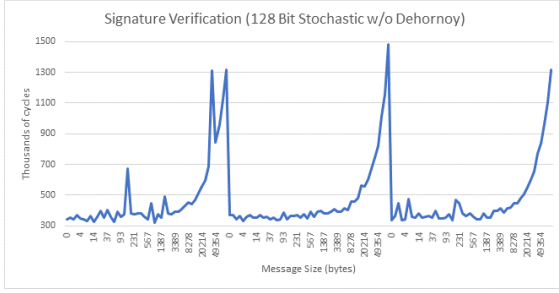


Fig. 13: SUPERCOP output for verifying 128-bit signatures with Stochastic Rewriting, without Dehornoy

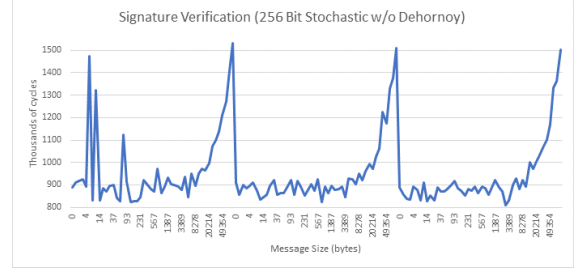


Fig. 14: SUPERCOP output for verifying 256-bit signatures with Stochastic Rewriting, without Dehornoy

Figure 9 shows clearly that for smaller messages the mathematical computation dominates, but the hash computation starts to dominate once input messages reach about 8000 bytes. Specifically, looking at Table 3, it appears that the run time starts to increase once messages reach somewhere between 1500-4000 bytes, and the hash function dominates, more than doubling the execution time, between 16000-20000 bytes. A significantly optimized hash implementation is clearly a requirement.

Message Size (B)	567	709	887	1109	1387	1734	2168	2711	3389	4237	5297	6622
Cycles	266344	248603	234803	236032	233528	241642	265536	243007	266480	290505	310513	316516
Message Size (B)	8278	10348	12936	16171	20214	25268	31586	39483	49354	61693	77117	96397
Cycles	308187	339611	541637	374523	424190	487219	518486	632247	734413	820023	997092	1217209

Table 3: SUPERCOP cycle counts for signature verification

The same pattern, where message size affects verification speed, can also be seen for 256-bit signatures in Figure 10. Similarly, Figures 11, 12, 13, and 14 all exhibit the same structure. The main difference between them all is the baseline computation time. The graphs show that Dehornoy vs non-Dehornoy is the main component to speed (which is to be expected, considering validation computation time is linear in the length of the signature, and Dehornoy reduces the signature size). Still, without Dehornoy the base verification time is around 350000 cycles for 128-bit signatures and about 875000 cycles for 256-bit signatures, and when Dehornoy is applied those times are reduced to 200000 and 500000 cycles.

In addition to working on the target platform with SUPERCOP, we also took an average-length signature at 128-bit security level and ran the verification computation on various embedded processors used on devices associated with the Internet of Things. See Table 4 for the raw data.

C Expected Security Strength

There are several known exponential attacks against WalnutDSA (see appendix E). In other cases the best-known attack is brute-force. See [1] for a full security analysis. The summary is that for a given security level (SL):

Platform	Bits	Clock (MHz)	ROM	RAM	Cycles	Time (ms)
ARM Cortex R5	32	200	2024	1060	483,210	2.4
ARM Cortex M4	32	168	4512	1116	817,545	4.866
RISC-V FE310	32	256	n/a	n/a	2,207,010	8.62
ARM Cortex M3	32	48	4568	1116	1,001,829	20.89
ARM Cortex M0	32	32	4524	1156	3,262,152	101.6
RL78	16	32	3830	1116	4,215,741	131.7

Table 4: Raw WalnutDSA performance data for verifying 128-bit signatures

1. A public key is a pair of $N \times N$ matrices of elements in F_q , which implies there are a maximum of q^{N^2} potential options per matrix. However, due to the construction, this is reduced to a minimum of $q^{N(N-3)}$ potential matrices. We have chosen N and q such that this value exceeds 2^{SL} possible public keys.
Moreover, to defeat a Pollard-rho style attack, we need to ensure that $q^{N(N-3)-1} > 2^{2SL}$. We have chosen N and q such that this inequality holds.
2. A secret key is a pair of braidwords in B_N of length ℓ . The number ℓ is chosen such that there are at least 2^{SL} unique braids when randomly creating a braid of length ℓ (prior to free reduction).
3. Concealed cloaking elements are chosen and inserted randomly into the cloaking elements (v_1 , v_2 , and v_3) and private keys to ensure sufficient protection against [11], reducing it to a brute-force search. Specifically, we ensure a brute-force level of $(N!)^\kappa > 2^{\text{SecurityLevel}}$ for all parameter levels because $\kappa = 6\kappa_1 + 2\kappa_2$. Collisions in the hash or in these random permutations do not help an attacker, so no birthday attack can be leveraged.
4. Reversing the rewriting schemes is also a brute-force problem which far exceeds 2^{SL} operations to reverse.
5. The message encoding parameters were chosen to ensure that the computed dimension of the resulting vector space is high enough such that the inequality $q^{\text{dimension}} > 2^{2SL}$ holds.
6. The choices of N and q are further defined to ensure that E -Multiplication is a one-way function and that reversing it requires $q^{(N-1)/2}\sqrt{60}$ run time⁶.

We have increased our values for κ and ℓ by 25% beyond the minimum required to meet the desired security level, both for 128- and 256-bit (conventional) security. This increase is purely for future proofing against minor errors or miscalculations in the number of possible braid or braidwords or improvements in enumeration techniques.

We believe that WalnutDSA is subject to Grover so we expect that the quantum security is half of the conventional security.

D Advantages and Limitations

The main advantages of WalnutDSA are that key generation and signature validation are extremely fast, even on small, constrained devices. These functions can be implemented in very little code and compile down to very small targets. Indeed, the raw signature validation on a 48MHz ARM Cortex M3 can complete in 5.7ms in compiled C software. Due to its nature,

⁶ The factor $\sqrt{60}$ was provided by the authors of the attack as a parameter they set

signature validation can easily be computed even on 16- or 8-bit processors with limited RAM and ROM and decent performance.

The main limitation of WalnutDSA is that signature generation is more expensive because the known braid rewriting techniques are more computationally intensive.

E Known Attacks

If $\text{Priv}(S) = \text{Priv}(S')$ then there is a factoring attack that can potentially create a valid signature by combining multiple signatures to create new words. However, the signatures generated by this attack are orders of magnitude longer than a valid WalnutDSA signature (estimated at a length 2^{32} or longer), and this attack is completely defeated by ensuring that $\text{Priv}(S) \neq \text{Priv}(S')$. See [1] for an analysis of this attack.

In January, 2018, [4] noted that the factoring attack could still succeed even when $\text{Priv}(S) \neq \text{Priv}(S')$ with a resulting forged signature double the length of the original attack. The original attacks resulted in forgeries well greater than the 2^{14} limit imposed by this specification, and the smallest forgery they conjecture would still result in signatures that are 2^7 times longer than the allowed limit. Further reductions of the forged signatures by e.g. [8] only result in an additional 10% gain, still resulting in signatures well beyond the allowed limit.

Also in January, 2018, [7] showed a Pollard-rho style attack which showed that to reach a desired security level of k -bits, $q^{N(N-3)-1} > 2^{2k}$. This inequality is satisfied with the parameters in this specification.

In February, 2018, [5] found that the original WalnutDSA encoding parameter (how to convert the hash result to a braid) was not injective. He also noted that the vector space generated by the encoding required a dimension high enough to reach the desired security level, specifically, to reach a desired security level of k -bits, $q^{\text{dimension}} > 2^{2k}$. For a parameter of $N = 10$ and the message encoding parameters in this specification, the dimension can easily be determined to be 66 when $\tau_1 = \tau_2 = 1$, and 82 when τ_1 is random and $\tau_2 = -\tau_1^{-1}$, which satisfies this inequality for the parameters listed in this specification.

In April, 2018, [6] found another exponential factoring attack that generates short replacement private keys. The original code they provided was proven to run in time $q^{N-5/2}$. They claim they reduced this to $q^{N/2-1}$ but the updated code they supplied failed in tests with simple inputs like B_3F_7 more than half the time, sometimes taking well over 12 hours to solve one input, such that this runtime could not be validated. However, assuming this $q^{N/2-1}$ is correct, a simple parameter change to $N=11$, $q=2^{31}-1$ meets this requirement for 128-bit security, and $N=11$, $q=2^{61}-1$ for 256-bit security.

This run time assumes $\tau_1 = \tau_2 = 1$, however, with the change to a prime field, additional methods to create cloaking elements become available which no longer requires T -values to be 1. With this change, the theoretical runtime for this attack increases⁷ to $q^{(N-1)/2}\sqrt{60}$, which allows decreasing N to 10 and maintaining the 128- and 256-bit security levels.

In May, 2018, [11] found a braid-based attack that found and removed the cloaking elements in order to determine the secret braids. This attack required cloaking elements be conjugates and also knowledge of the cloaked permutation. Adding the concealed cloaking elements inside the left and right sides of v_1 , v_2 , and v_3 makes them no longer conjugates,

⁷ The $\sqrt{60}$ factor was supplied by the authors of the attack, focused on B_8 , and may well be larger for $N=10$.

and with the unknown permutations it requires searching through $N!$ permutations for each concealed cloaking element. By adding a total of κ concealed cloaking elements it requires $(N!)^\kappa$ work, which can be made sufficiently large to ensure 128- and 256- bit security levels.

The next best-known attack against WalnutDSA is a brute force search. See [1] for a full security analysis, including a proof of security under EUF-CMA.

There are no known other attacks against WalnutDSA as of this writing.